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RATIONALIZING POLICY FUNCTIONS BY DYNAMIC OPTIMIZATION

BY TAPAN MITRA AND GERHARD SORGER[1]

We derive necessary and sufficient conditions for a pair of functions to be the optimal policy function and the optimal value function of a dynamic maximization problem with convex constraints and concave objective functional. It is shown that every Lipschitz continuous function can be the solution of such a problem. If the maintained assumptions include free disposal and monotonicity, then we obtain a complete characterization of all optimal policy and optimal value functions. This is the case, e.g., in the standard aggregative optimal growth model.

KEYWORDS: Dynamic optimization, optimal growth theory, rationalizability.

1. INTRODUCTION

THIS PAPER STUDIES THE SET OF SOLUTIONS of dynamic optimization problems of the form

$$V(x) = \sup \sum_{t=0}^{+\infty} \rho^t U(x_t, x_{t+1})$$

$$\text{subject to} \qquad (x_t, x_{t+1}) \in \Omega \qquad \text{and} \qquad x_0 = x.$$

Here, ρ is the discount factor, U is the utility function, Ω describes the transition possibility set, and x is the initial state of the system. Models of this form arise in many different areas of economics, notably in optimal growth theory (see Stokey and Lucas (1989) and McKenzie (1986)). Under standard convexity and continuity assumptions one can show that the optimal paths of this problem (starting at any initial state x) are characterized by a continuous function h which maps the state at time t, x_t, to its unique optimal successor state $x_{t+1} = h(x_t)$. The solution of the above problem is therefore compactly described by the pair (h, V) consisting of the optimal policy function h and the optimal value function V, respectively. Conversely, one can say that the pair (h, V) is rationalized by the dynamic optimization problem (Ω, U, ρ).

The goal of this paper is to characterize the set of all pairs (h, V) that can be rationalized by a problem (Ω, U, ρ) satisfying the standard convexity and continuity assumptions. Similar characterizations of the solution sets for other standard models in economic theory have gained considerable attention in the literature. The most prominent examples are probably the characterization of individual demand functions by the strong axiom of revealed preference (see Samuelson (1938, 1947), Houthakker (1950), and Uzawa (1960)) and the charac-

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terization of market excess demand functions by continuity, homogeneity of degree 0, and Walras' law (see Sonnenschein (1973, 1974), Mantel (1974), and Debreu (1974)).

As for dynamic optimization problems of the form described above, previous research has focussed entirely on the characterization of the set of optimal policy functions h without considering the optimal value function V as part of the solution. From the maximum theorem it follows that every optimal policy function h has to be continuous. The first step towards a sufficient rationalizability condition was taken by Boldrin and Montrucchio (1986) who showed that any twice continuously differentiable function can be an optimal policy function. Neumann et al. (1988) slightly relaxed this condition by proving that any differentiable function with a Lipschitz continuous derivative can be an optimal policy function.[2] In the same paper, the authors also provided an example of an optimal policy function that is not Lipschitz continuous. Hewage and Neumann (1990), finally, showed that not all continuous functions can be optimal policy functions in a dynamic optimization problem satisfying the standard assumptions.[3] To summarize, the state of knowledge is that every optimal policy function is continuous, that there are continuous functions that are not rationalizable, that Lipschitz continuity is not necessary for rationalizability, and that differentiability with a Lipschitz continuous derivative is sufficient for rationalizability.

In the present paper we present conditions for the rationalizability of a pair (h, V) consisting of both the policy and the value function. The necessary and sufficient conditions that we derive are very close to each other. The necessary condition requires that both h and V are continuous, V is strictly concave, and, for every state x at which there exists a price (subgradient) p supporting V at x, one can also find a price q supporting V at $h(x)$ such that the inequality

$$V(h(x)) - V(h(y)) + q[h(y) - h(x)] \\ \leq (1/\rho)[V(x) - V(y) + p(y - x)]$$

holds for all states y. The sufficient condition requires in addition that V can be extended as a concave function to an open set containing the state space of the model. The above inequality is a joint condition on h and V and it says that either the distance between $h(x)$ and $h(y)$ cannot be too large as compared to the distance between x and y (which is related to Lipschitz continuity of h) or that the curvature of V between $h(x)$ and $h(y)$ has to be very small as compared to the curvature of V between x and y.

Using our tight conditions on the pair (h, V) we are able to reconsider the question of rationalizability of h and answer it in a much more satisfying way

[2]The analysis in Neumann et al. (1988) is restricted to the case of one-dimensional state spaces. See Montrucchio (1994) for a proof of this result in general multi-sector models.

[3]This result has been replicated under various assumptions by Sorger (1995) and Mitra (1996a). They all involve continuous functions which are infinitely steep at an interior fixed point.

than it was possible before. We considerably improve the results by Boldrin and Montrucchio (1986), Neumann et al. (1988), and Montrucchio (1994) by showing that any Lipschitz continuous function can be an optimal policy function. Coming back to the rationalizability problem for pairs (h, V) we note that the only difference between the necessary condition and the sufficient one is that in the latter we require that V can be extended as a concave function to some open set containing the state space. We demonstrate that one can dispense with this additional requirement if one replaces the continuity assumption on the utility function U by the assumption of boundedness and upper semi-continuity and if one assumes free disposal and certain monotonicity properties for Ω and U. In this slightly different framework, the rationalizability condition mentioned above is augmented by the monotonicity of V and the nonnegativity of the support prices p and q. We show that with these modifications the condition is both necessary and sufficient for the existence of a dynamic optimization problem (Ω, U, ρ) which is solved by (h, V). A particular example that fits this framework is the standard aggregative model of optimal growth.

Section 2 formulates the dynamic optimization problems under consideration. In Section 3 we state the general rationalizability conditions and prove that every Lipschitz continuous function can be rationalized. Section 4 considers the modified set of assumptions and derives a necessary and sufficient rationalizability condition in this framework. Section 5 presents concluding remarks.

2. DYNAMIC OPTIMIZATION PROBLEMS

Time is measured in discrete periods $t \in \{0,1,2,\dots\}$. At each time t the state of the economic system is described by a vector $x_t \in X$ where the state space $X \subseteq \mathbb{R}^n$ is a compact and convex set with nonempty interior. The dynamic optimization problem consists in finding

$$V(x) = \sup \sum_{t=0}^{+\infty} \rho^t U(x_t, x_{t+1}) \tag{1}$$

where the supremum is taken over the set of all sequences $(x_t)_{t=0}^{+\infty}$ satisfying the constraints

$$(x_t, x_{t+1}) \in \Omega, \qquad t \in \{0,1,2,\dots\}, \tag{2}$$

$$x_0 = x. \tag{3}$$

Here, ρ is the discount factor, U is the utility function, Ω is the constraint set, $x \in X$ is the initial state, and V is the optimal value function. We now discuss the assumptions that will be used in this paper (except for Section 4 where we introduce a different set of assumptions).

A1: *$\Omega \subseteq X \times X$ is a closed and convex set such that the x-section $\Omega_x = \{y \in X \mid (x, y) \in \Omega\}$ is nonempty for all $x \in X$. The set $\bigcup_{x \in X} \Omega_x$ has nonempty interior.*[4]

A2: *U: $\Omega \mapsto \mathbb{R}$ is a continuous and concave function.*

A3: $\rho \in (0,1)$.

We shall refer to the dynamic optimization problem (1)–(3) as problem (Ω, U, ρ). Note that we do not include the initial state x in the description of the problem. Thus, problem (Ω, U, ρ) requires finding the optimal state trajectories from *any* initial state $x \in X$. Assumptions A1–A3 are standard assumptions in the relevant literature and they imply that the Bellman equation

$$V(x) = \max\{U(x, y) + \rho V(y) \mid y \in \Omega_x\}$$

holds for all $x \in X$. Moreover, a path $(x_t)_{t=0}^{+\infty}$ satisfying (2) and (3) is optimal if and only if $V(x_t) = U(x_t, x_{t+1}) + \rho V(x_{t+1})$ holds for all $t \in \{0, 1, 2, \ldots\}$. However, optimal paths for (1)–(3) need not be unique. To ensure that optimal paths are unique one has to add strict concavity.

A4: *The optimal value function V is strictly concave.*

If A1–A4 hold, then the following is true. For every $x \in X$ there exists exactly one $y \in \Omega_x$ such that $V(x) = U(x, y) + \rho V(y)$. In other words, there exists a unique maximizer on the right-hand side of the Bellman equation. Let $h(x)$ denote this maximizer, that is, $h(x) = \text{argmax}\{U(x, y) + \rho V(y) \mid y \in \Omega_x\}$. The function h: $X \mapsto X$ defined in that way is called the optimal policy function of the optimization problem (Ω, U, ρ). It maps any state $x \in X$ to its optimal successor state $h(x)$. Optimal paths are uniquely determined as the trajectories of the difference equation $x_{t+1} = h(x_t)$ with (3) as the initial condition. We shall call the pair (h, V) the solution of the optimization problem (Ω, U, ρ) or we shall say that (Ω, U, ρ) rationalizes (h, V). Sometimes we shall restrict attention to the optimal policy function by saying that (Ω, U, ρ) rationalizes h. The following proposition summarizes necessary and sufficient optimality conditions for (Ω, U, ρ) under Assumptions A1–A4. For a proof we refer to Chapter 4 in Stokey and Lucas (1989).

PROPOSITION 1: *Let (Ω, U, ρ) be an optimization problem on the state space X satisfying $A1$–$A3$.* (a) *If problem (Ω, U, ρ) satisfies $A4$ and (h, V) is its solution, then the following conditions hold.*

C1: *The function h: $X \mapsto X$ is continuous and satisfies $h(x) \in \Omega_x$ for all $x \in X$.*

C2: *The function V: $X \mapsto \mathbb{R}$ is continuous and strictly concave.*

[4] The assumption that $\bigcup_{x \in X} \Omega_x$ has nonempty interior in X is satisfied whenever Ω has nonempty interior in $X \times X$. The converse is not true as can be seen by simple examples.

C3: *For all* $(x, y) \in \Omega$ *with* $y \neq h(x)$ *it holds that*

$$V(x) > U(x, y) + \rho V(y), \tag{4}$$

$$V(x) = U(x, h(x)) + \rho V(h(x)). \tag{5}$$

(b) *Conversely, if there exists a pair of functions* (h, V) *satisfying* $C1-C3$, *then it follows that* (Ω, U, ρ) *satisfies* $A4$ *and that* (h, V) *is its solution.*

Conditions C1 and C2 show that the solution of an optimization problem satisfying A1–A4 always consists of a pair of continuous functions. The purpose of the present paper is to characterize in more detail the set of all pairs (h, V) that can be rationalized by such optimization problems. Some of our results involve a concavity assumption that is based on the notion of α-concavity and α-convexity. If α is any real number, then V is α-concave if $x \mapsto V(x) + (\alpha/2)\|x\|^2$ is a concave function. Analogously, we say that V is α-convex if $x \mapsto V(x) - (\alpha/2)\|x\|^2$ is convex.[5]

A5: *There exist positive real numbers* α *and* β *such that* V *is* α-*concave and* $(-\beta)$-*convex.*

It is obvious that this assumption is stronger than A4. Sufficient conditions for α-concavity of V are stated, e.g., in Montrucchio (1987). Sufficient conditions for $(-\beta)$-convexity of V are harder to obtain and would certainly involve nontrivial joint restrictions on both U and Ω. In many cases (especially in optimal growth theory) the optimization problem (Ω, U, ρ) is also assumed to satisfy the following monotonicity assumption.

A6: *If* $x \leq \bar{x}$, *then* $\Omega_x \subseteq \Omega_{\bar{x}}$. *The function* $x \mapsto U(x, y)$ *is nondecreasing and the function* $y \mapsto U(x, y)$ *is nonincreasing.*

It is known that an optimization problem (Ω, U, ρ) which satisfies A1–A4 and A6 has a nondecreasing optimal value function V (see, e.g., Theorem 4.7 in Stokey and Lucas (1989)).

3. CONDITIONS FOR RATIONALIZABILITY

Let X denote the state space and let ρ be a number in $(0, 1)$. In the first theorem we shall demonstrate that the following condition is necessarily satisfied if (h, V) is the solution of an optimization problem (Ω, U, ρ) satisfying A1–A4.

[5]For a discussion of these concepts we refer to Vial (1983). Some authors (e.g. Montrucchio (1994)) use the term concavity-β instead of $(-\beta)$-convexity.

R(ρ): *The functions h: $X \mapsto X$ and V: $X \mapsto \mathbb{R}$ are continuous and V is strictly concave. For all $x \in X$ such that $\partial V(x) \neq \emptyset$ there exist $p_x \in \partial V(x)$ and $q_x \in \partial V(h(x))$ such that*

$$(6) \qquad V(h(x)) - V(h(y)) + q_x[h(y) - h(x)] \leq (1/\rho)[V(x) - V(y) + p_x(y-x)]$$

holds for all $y \in X$.[6]

Note that the qualification $\partial V(x) \neq \emptyset$ in condition R(ρ) holds necessarily for all $x \in \text{int } X$ but may or may not hold at boundary points of the state space.

THEOREM 1: *Let $X \subseteq \mathbb{R}^n$ be a compact and convex set with nonempty interior and h: $X \mapsto X$ and V: $X \mapsto \mathbb{R}$ two given functions. If the pair (h,V) can be rationalized by a dynamic optimization problem (Ω, U, ρ) on X such that Assumptions $A1$–$A4$ hold, then (h,V) must satisfy* R(ρ).

PROOF: Assume that (h,V) is the solution of (Ω, U, ρ). Continuity of h and V and strict concavity of V are implied by Proposition 1a. It remains to prove (6). Consider any state $x \in X$ and any subgradient $p_x \in \partial V(x)$. By a well known separation argument there exists $q_x \in \partial V(h(x))$ such that

$$U(x,h(x)) + \rho q_x h(x) - p_x x \geq U(y,z) + \rho q_x z - p_x y$$

for all $(y,z) \in \Omega$.[7] For $z = h(y)$ one gets

$$U(x,h(x)) + \rho q_x h(x) - p_x x \geq U(y,h(y)) + \rho q_x h(y) - p_x y.$$

From Proposition 1a it follows that $U(x,h(x)) = V(x) - \rho V(h(x))$ and $U(y,h(y)) = V(y) - \rho V(h(y))$. Substituting this into the above inequality and rearranging terms leads to (6). *Q.E.D.*

To explain the basic idea of the proof of Theorem 1 in simple geometric terms it is convenient to assume that (h,V) is rationalized by a model (Ω, U, ρ) with a continuously differentiable utility function U and that $h(x) \in \text{int } \Omega_x$ for all $x \in X$. In this case the optimal value function V is also continuously differentiable on the interior of X and the first order and envelope conditions

$$(7) \qquad U_1(x,h(x)) = V'(x), \qquad U_2(x,h(x)) = -\rho V'(h(x))$$

must hold (see Theorem 4.11 in Stokey and Lucas (1989)). Consider any pair (x,y) of different states. From (5) we know that

$$(8) \qquad U(x,h(x)) = V(x) - \rho V(h(x)), \qquad U(y,h(y)) = V(y) - \rho V(h(y)).$$

[6]$\partial V(z)$ denotes the subdifferential of V at z, that is, $\partial V(z) = \{p \in \mathbb{R}^n \mid V(y) \leq V(z) + p(y-z)$ for all $y \in \mathbb{R}^n\}$.

[7]The argument was first presented by Weitzman (1973) using assumptions that are different from ours. A proof that is valid under the present assumptions can be found in McKenzie (1986, pp. 1288–1289).

Concavity of U implies that $U(y, h(y))$ lies below the tangent hyperplane that supports U at the point $(x, h(x))$. Using (7) and (8) this property translates into $V(y) - \rho V(h(y)) \leq V(x) - \rho V(h(x)) + V'(x)(y - x) - \rho V'(h(x))[h(y) - h(x)]$, which is equivalent to (6) under the assumption that V is differentiable. In the case where V is not differentiable the first order and envelope conditions (7) need not hold in this form and one has to replace the usual gradients by subgradients and to restrict the validity of (6) to those pairs $(x, y) \in X \times X$ for which $\partial V(x)$ is nonempty.

To interpret condition R(ρ) it is useful to define $d_V(x, y) = V(x) - V(y) + p_x(y - x)$. With this notation R($\rho$) can be restated as $d_V(h(x), h(y)) \leq (1/\rho) d_V(x, y)$. Now note that strict concavity of V implies $d_V(x, y) > 0$ for all $(x, y) \in X \times X$ with $x \neq y$ and $d_V(x, x) = 0$ for all $x \in X$. Although d_V is not a metric (in general, it is neither symmetric nor does it satisfy the triangle inequality), one may think of $d_V(x, y)$ as measuring the distance between the states x and y. For example, it is easily seen that for $z \neq 0$ the function $g(\lambda) = d_V(x, x + \lambda z)$ is strictly convex, strictly increasing, and nonnegative with respect to $\lambda \in [0, +\infty)$, and attains its unique minimum $g(0) = 0$. Note that, as λ increases, the point $x + \lambda z$ moves away from x in the direction z and, consequently, the 'distance' $d_V(x, x + \lambda z)$ increases. However, the value of $d_V(x, y)$ depends also on the curvature of V. For example, if V is twice continuously differentiable, then one has

$$d_V(x, y) = -(1/2)(y - x)^T V''(z)(y - x)$$

for some point z on the line segment joining x and y. This shows that if the slope of V changes sharply as one goes from x to y (which means that the eigenvalues of $V''(z)$ are very small), then $d_V(x, y)$ will be large. Keeping in mind the "distance and curvature" interpretation of $d_V(x, y)$, condition $R(\rho)$ says that either $h(x)$ is not too far away from $h(y)$ as compared to the distance between x and y, or the curvature of V between $h(x)$ and $h(y)$ is small as compared to the curvature of V between x and y.

We now turn to the following sufficient condition for the rationalizability of a pair (h, V).

R*(ρ): *The functions h: $X \mapsto X$ and V: $X \mapsto \mathbb{R}$ are continuous and V is strictly concave. There exists an open and convex set X^* containing X and a concave function V^*: $X^* \mapsto \mathbb{R}$ which coincides with V on X such that the following is true: for every $x \in X$ there exist subgradients $p_x \in \partial V^*(x)$ and $q_x \in \partial V^*(h(x))$ such that* (6) *holds for all $y \in X$.*

Note that $\partial V^*(x)$ and $\partial V^*(h(x))$ are nonempty because both x and $h(x)$ are elements of X and, thus, in the interior of X^*. The difference between R*(ρ) and R(ρ) is that in the former we require that V can be extended as a concave function to an open set containing X. Thereby, we rule out the existence of states $x \in X$ at which the subdifferential $\partial V^*(x)$ becomes empty or unbounded.

THEOREM 2: *Let $X \subseteq \mathbb{R}^n$ be a compact and convex set with nonempty interior and h: $X \mapsto X$ and V: $X \mapsto \mathbb{R}$ two given functions. If there exists $\rho \in (0,1)$ such that condition* R*(ρ) *is satisfied, then one can rationalize (h,V) by a dynamic optimization problem (Ω, U, ρ) satisfying A1–A4. If, in addition, V is nondecreasing, then (Ω, U, ρ) can be chosen such that A6 holds.*

PROOF: Let (h,V) be given such that R*(ρ) holds. To construct an optimization problem (Ω, U, ρ) which rationalizes (h,V) we proceed in three steps. First we construct (Ω, U, ρ) and prove that Assumptions A1–A3 are satisfied. Then we verify conditions C1–C3 of Proposition 1. The rationalizability of (h,V) and the validity of A4 are then an immediate consequence of Proposition 1b. Finally, we show that A6 is satisfied if V is nondecreasing.

STEP 1: Let $\Omega = X \times X$; then Assumption A1 is satisfied. For every $z \in X$ define the function F_z: $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ by

(9) $$F_z(x,y) = V(z) - \rho V(h(z)) + p_z(x-z) - \rho q_z[y - h(z)]$$

where p_z and q_z are the subgradients of V^* whose existence is ensured by R*(ρ). Moreover, define $U(x,y) = \inf\{F_z(x,y) \mid z \in X\}$. From the fact that X is a compact subset of the interior of X^* it follows that $F_z(x,y)$ is uniformly bounded from below on any compact set $A \subseteq \mathbb{R}^n \times \mathbb{R}^n$, that is, there exists a finite number $M(A)$ such that $F_z(x,y) \geq M(A)$ for all $z \in X$ and all $(x,y) \in A$. Obviously, this implies that U is a finite function, that is, U: $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$. Moreover, each of the functions F_z is affine so that U (as the infimum of a family of affine functions) is a concave function. Since U is finite and concave on $\mathbb{R}^n \times \mathbb{R}^n$ it follows that U is continuous. We have therefore shown that A2 holds. Assumption A3 holds trivially.

STEP 2: Continuity of h and V as well as strict concavity of V hold by assumption R*(ρ). The property $h(x) \in \Omega_x$ for all $x \in X$ holds by definition of Ω. Thus C1 and C2 of Proposition 1 are satisfied. It remains to verify C3. Let $x \in X$ be given. For any $y \in \Omega_x$ such that $y \neq h(x)$ we have

$$\begin{aligned} U(x,y) &\leq F_x(x,y) = V(x) - \rho V(h(x)) - \rho q_x[y - h(x)] \\ &< V(x) - \rho V(y). \end{aligned}$$

Here, the first step follows from the definition of U, the second step from the definition of F_x, and the third one from strict concavity of V and $y \neq h(x)$. Thus, (4) is verified. Now observe that for all $x \in X$ we have

(10) $$V(x) - \rho V(h(x)) = F_x(x, h(x)).$$

Condition R*(ρ) implies

$$\begin{aligned} &V(h(z)) - V(h(x)) + q_z[h(x) - h(z)] \\ &\quad \leq (1/\rho)[V(z) - V(x) + p_z(x-z)] \end{aligned}$$

for all $z \in X$. Obviously, this inequality is equivalent to

$$(11) \qquad F_x(x,h(x)) \leq F_z(x,h(x)).$$

Since $z \in X$ was arbitrary we obtain from (10) and (11) that $V(x) - \rho V(h(x)) = F_x(x, h(x)) = \inf\{F_z(x, h(x)) \mid z \in X\} = U(x, h(x))$. Thus, (5) is verified and Proposition 1 yields the result.

STEP 3: To see that A6 can be satisfied whenever V is nondecreasing observe that monotonicity of V implies that for all $z \in X$ the subgradients p_z and q_z are nonnegative. Thus, $F_z(x, y)$ defined in Step 1 is nondecreasing in x and nonincreasing in y for all z. Obviously, these properties are inherited by $U(x, y)$. *Q.E.D.*

The crux in the constructive proof of Theorem 2 is finding the function U. In order to explain the geometric intuition of this step, first note that for any model (Ω, U, ρ) which possibly rationalizes (h, V) it must hold that $U(x, h(x)) = V(x) - \rho V(h(x))$ (see Equation (5)). Thus, the value of the utility function is determined by (h, V) for every point on the graph of h. If U were differentiable and $h(x) \in \operatorname{int} \Omega_x$, then the first order and envelope conditions (7) would hold and the tangent hyperplanes of U at any point on the graph of h would also be determined by (h, V). The basic idea of the proof of Theorem 2 is to define the graph of U as the envelope of all these hyperplanes. The main technical difficulty with this approach is that V cannot be assumed to be differentiable and, therefore, the first order and envelope conditions (7) may not hold. Therefore, we have to replace the derivatives of V by subgradients and we have to assume that V can be extended as a concave function to some open set containing X. Whereas using subgradients does not introduce any gap between the necessary condition R(ρ) and the sufficient condition R*(ρ), the extendability of V does. In this regard it is worth pointing out that R*(ρ) is certainly not necessary for the rationalizability of (h, V) since we know that there are examples of rationalizable pairs (h, V) for which V cannot be extended as a concave function to an open set containing X. This is the case, for example, if V is infinitely steep at the boundary of X. In Section 4 we study what can be said if one drops the additional assumption of extendability of V.

Perhaps the most interesting consequence of Theorem 2 is that every Lipschitz continuous function h can be the optimal policy function of a dynamic optimization problem satisfying A1–A6.

THEOREM 3: *Let $X \subseteq \mathbb{R}^n$ be a compact and convex set with nonempty interior and let h: $X \mapsto X$ be a Lipschitz continuous function with Lipschitz constant L. For every $\rho \leq 1/L^2$ there exists an optimization problem (Ω, U, ρ) satisfying A1–A6 which rationalizes h.*

PROOF: To prove the result define the quadratic function $V: \mathbb{R}^n \mapsto \mathbb{R}$

(12) $$V(x) = \gamma e_n x - (\alpha/2)\|x\|^2,$$

where α and γ are positive real numbers and $e_n = (1,1,\ldots,1) \in \mathbb{R}^n$. Because X is compact, V is strictly increasing on X provided γ is chosen sufficiently large. Moreover, V is both α-concave and $(-\alpha)$-convex and it is continuously differentiable on $\mathbb{R}^n$. These properties imply that A5 (and, hence, A4) holds and that one may choose $V^* = V$ in condition R*(ρ). Using the property $\partial V(x) = \{p_x\}$ with $p_x = \gamma e_n - \alpha x$ one obtains

$$\begin{aligned} V(x) - V(y) + p_x(y-x) &= \gamma e_n x - (\alpha/2)\|x\|^2 - \gamma e_n y \\ &\quad + (\alpha/2)\|y\|^2 + (\gamma e_n - \alpha x)(y-x) \\ &= (\alpha/2)\|x-y\|^2. \end{aligned}$$

In a similar way (using the fact that $\partial V(h(x)) = \{q_x\}$ with $q_x = \gamma e_n - \alpha h(x)$) one can show that

$$V(h(x)) - V(h(y)) + q_x[h(y) - h(x)] = (\alpha/2)\|h(x) - h(y)\|^2.$$

Condition (6) is therefore equivalent to $\|h(x) - h(y)\|^2 \leq (1/\rho)\|x-y\|^2$. This inequality holds for all $(x,y) \in X \times X$ if the discount factor ρ is chosen such that $\rho \leq 1/L^2$. The result is therefore an immediate consequence of Theorem 2. *Q.E.D.*

The converse of this theorem is well known: every dynamic optimization problem satisfying A1–A5 has a Lipschitz continuous optimal policy function (see, e.g., Montrucchio (1987, 1994) or Sorger (1994, 1995) for proofs under various assumptions). For this implication, however, the strong concavity assumption A5 is crucial, since there are optimal policy functions of dynamic optimization problems satisfying A1–A4 (but not A5) that are not Lipschitz continuous (e.g. Example 2 in Neumann et al. (1988)). We illustrate the application of Theorem 2 by providing an example in which the optimal policy function does not have finite steepness even at an interior fixed point.

EXAMPLE 1: Let $X = [-1,1]$, and define the pair (h,V) by

$$h(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ -\sqrt{x} & \text{if } x > 0, \end{cases} \qquad \text{and} \qquad V(x) = \begin{cases} 2x - x^4 & \text{if } x \leq 0, \\ 2x - x^2 & \text{if } x > 0. \end{cases}$$

Note that h is continuous but not Lipschitz continuous because it has slope $-\infty$ at its unique fixed point $x = 0 \in \operatorname{int} X$. Note also that V is continuous, strictly concave, and strictly increasing. We shall now show that condition R*(ρ) is satisfied for all $\rho \leq 1/3$. To this end first note that V is continuously differentiable on all of $\mathbb{R}$ such that the subgradients in (6) may be replaced by the usual derivatives and V^* may be chosen to be equal to V. We have to consider four different cases.

CASE 1 ($x \le 0$ and $y \le 0$): In this case we have $h(x) = h(y) = 0$ such that the left-hand side of (6) equals 0. Thus, (6) holds independently of ρ.

CASE 2 ($x \le 0$ and $y > 0$): In this case we have $h(x) = 0$ and $h(y) = -\sqrt{y} < 0$. This yields $V(x) - V(y) + V'(x)(y-x) = 3x^4 - 4x^3y + y^2$ and $V(h(x)) - V(h(y)) + V'(h(x))[h(y) - h(x)] = y^2$. Thus, (6) is satisfied provided that $\rho < (3x^4 - 4x^3y + y^2)/y^2$. Because $x \le 0$ and $y > 0$, the right-hand side of this inequality is greater than or equal to $(3x^4 + y^2)/y^2 \ge 1$ and we conclude that (6) holds in this case for all $\rho \in (0,1)$.

CASE 3 ($x > 0$ and $y \le 0$): In this case we have $h(x) = -\sqrt{x} < 0$ and $h(y) = 0$. This yields $V(x) - V(y) + V'(x)(y-x) = x^2 - 2xy + y^4$ and $V(h(x)) - V(h(y)) + V'(h(x))[h(y) - h(x)] = 3x^2$. Thus, (6) is satisfied provided that $\rho \le (x^2 - 2xy + y^4)/(3x^2)$. Because $x > 0$ and $y \le 0$, the right-hand side of this inequality is greater than or equal to $(x^2 + y^4)/(3x^2) \ge 1/3$ and we conclude that (6) holds in this case for all $\rho \in (0, 1/3]$.

CASE 4 ($x > 0$ and $y > 0$): In this case we have $h(x) = -\sqrt{x} < 0$ and $h(y) = -\sqrt{y} < 0$. This yields $V(x) - V(y) + V'(x)(y-x) = (x-y)^2$ and $V(h(x)) - V(h(y)) + V'(h(x))[h(y) - h(x)] = 3x^2 - 4x\sqrt{xy} + y^2$. Thus, (6) is satisfied provided that $\rho \le (x-y)^2/(3x^2 - 4x\sqrt{xy} + y^2)$. Because both x and y are positive and $x \ne y$ we have

$$\frac{(x-y)^2}{3x^2 - 4x\sqrt{xy} + y^2} = \frac{\left(\sqrt{x} - \sqrt{y}\right)^2\left(x + 2\sqrt{xy} + y\right)}{\left(\sqrt{x} - \sqrt{y}\right)^2\left(3x + 2\sqrt{xy} + y\right)}$$
$$= 1 - \frac{2x}{3x + 2\sqrt{xy} + y} \ge 1 - \frac{2}{3} = \frac{1}{3}.$$

Thus, (6) holds in this case for all $\rho \in (0, 1/3]$.

These results prove that (h, V) satisfies $R^*(\rho)$ whenever $\rho \in (0, 1/3]$ and we conclude from Theorem 2 that h can be rationalized by an optimization problem satisfying A1–A4 and A6. *Q.E.D.*

4. A COMPLETE CHARACTERIZATION

We have already pointed out that the gap between the necessary condition $R(\rho)$ and the sufficient condition $R^*(\rho)$ arises because the optimal value function may have empty subdifferential at boundary points of the state space. In a first attempt to prove Theorem 2 with condition $R^*(\rho)$ replaced by condition $R(\rho)$, one may be led to define $U(x,y)$ as the infimum of $F_z(x,y)$ where z ranges only over the interior of X (see (9) for the definition of $F_z(x,y)$). It is possible to show that this leads indeed to a utility function U that is bounded, concave, and upper semi-continuous but not necessarily continuous on the closed convex hull of the graph of h. Without continuity of U, however, the dynamic programming results stated in Proposition 1 need not hold. In the present section we introduce a different set of assumptions that does not include continuity of U but still ensures that the conditions stated in Proposition 1 hold.

Throughout this section we assume that the state space X is a nonempty, compact, and convex subset of $\mathbb{R}^n_+$ such that the following is true: if $\bar{x} \in X$, $x \in \mathbb{R}^n_+$, and $x \leq \bar{x}$, then $x \in X$.[8]

B1: *$\Omega \subseteq X \times X$ is a closed and convex set containing* $(0,0)$ *such that the following is true*: *if* $(x,y) \in \Omega$, $\bar{x} \in X$, $\bar{x} \geq x$, *and* $0 \leq \bar{y} \leq y$, *then* $(\bar{x}, \bar{y}) \in \Omega$.

B2: U: $\Omega \mapsto \mathbb{R}$ *is a bounded, concave, and upper semi-continuous function such that the following is true*: *if* $(x,y) \in \Omega$, $\bar{x} \in X$, $\bar{x} \geq x$, *and* $0 \leq \bar{y} \leq y$, *then* $U(\bar{x}, \bar{y}) \geq U(x,y)$.

The essential difference between these assumptions and the corresponding Assumptions A1 and A2 is that U is merely assumed to be bounded and upper semi-continuous instead of continuous, and that Ω and U are required to satisfy free disposal and monotonicity properties.[9] It is worth pointing out that $U(x_t, x_{t+1})$ is often interpreted as the "reduced" utility obtained by maximizing a "primitive" utility function $u(c_t)$ over consumption c_t subject to a given capital stock at the beginning of the period, x_t, and a fixed target capital stock at the end of the period, x_{t+1}. In this case it is possible to prove that U is upper semi-continuous, but it requires additional assumptions to justify continuity of U (see Dutta and Mitra (1989a)). This shows that in certain cases Assumptions B1 and B2 may be more appropriate than Assumptions A1 and A2.

It has been proved by Dutta and Mitra (1989b) that the optimal value function V is well defined, continuous, and concave and that the Bellman equation remains true if A1 and A2 are replaced by B1 and B2. In addition, it follows that V is nondecreasing. If one adds A4 then Proposition 1 can also be established with A1 and A2 replaced by B1 and B2. For the purpose of later reference we state this as a formal result. For a proof we refer to Dutta and Mitra (1989b).

PROPOSITION 2: *Let* (Ω, U, ρ) *be an optimization problem on* X *satisfying* $B1$, $B2$, *and* $A3$. (a) *If problem* (Ω, U, ρ) *satisfies* $A4$ *and* (h, V) *is its solution, then the following conditions hold*:

C1: *The function* h: $X \mapsto X$ *is continuous and satisfies* $h(x) \in \Omega_x$ *for all* $x \in X$.

C2: *The function* V: $X \mapsto \mathbb{R}$ *is continuous, strictly concave, and nondecreasing*.

C3: *For all* $(x,y) \in \Omega$ *and* $y \neq h(x)$ *conditions* (4) *and* (5) *hold*.

(b) *Conversely, if there exists a pair of functions* (h, V) *satisfying* $C1-C3$, *then it follows that* (Ω, U, ρ) *satisfies* $A4$ *and that* (h, V) *is its solution*.

[8] Such a set is called comprehensive. Note that $0 \in X$ holds for every comprehensive set X.

[9] Note that B1 and B2 imply A6 but that A6 does not contain any free disposal assumption.

The only difference between the conclusions of Proposition 1 and Proposition 2 is that condition C2 now requires V to be nondecreasing. This implies that V has nonnegative subgradients whenever it is subdifferentiable. The rationalizability condition that we are using in the present section is therefore a slight modification of R(ρ) which takes this fact into account.

R$^+$(ρ): *The functions h: $X \mapsto X$ and V: $X \mapsto \mathbb{R}$ are continuous and V is strictly concave and nondecreasing. For all $x \in X$ such that $\partial V(x) \neq \emptyset$ there exist $p_x \in \partial V(x)$ and $q_x \in \partial V(h(x))$ such that $p_x \geq 0$, $q_x \geq 0$, and such that* (6) *holds for all $y \in X$.*

THEOREM 4: *Let $X \subseteq \mathbb{R}^n_+$ be a compact, convex, and comprehensive set with nonempty interior. Moreover, let h: $X \mapsto X$ and V: $X \mapsto \mathbb{R}$ be two given functions.*

(a) *If (h, V) can be rationalized by an optimization problem (Ω, U, ρ) on X such that $B1$, $B2$, $A3$, and $A4$ are satisfied and such that there exists $(x, y) \in \Omega$ with $y \gg 0$, then condition $R^+(\rho)$ holds.*[10]

(b) *If there exists $\rho \in (0,1)$ such that (h, V) satisfies $R^+(\rho)$, then one can rationalize (h, V) by a dynamic optimization problem satisfying $B1$, $B2$, $A3$, and $A4$.*

PROOF: (a) The proof is almost identical to the one of Theorem 1. Note that the separation argument guaranteeing the existence of the subgradient q_x does not rely on the continuity of the utility function. The requirement that there exists $(x, y) \in \Omega$ with $y \gg 0$ replaces the assumption that $\bigcup_{x \in X} \Omega_x$ has nonempty interior (see A1). The fact that V is nondecreasing allows one to deduce that both p_x and q_x are nonnegative vectors.

(b) The general idea of the proof is the same as for Theorem 2 (only Step 1 and Step 2 are required). The details are a little bit different, though.

STEP 1: Let D be the convex hull of the graph of h. Since X is compact and h is continuous, it is straightforward to show that D is compact. Now define

$$\Omega = \{(x, y) \mid x \in X,\ y \in X,\ \exists (\bar{x}, \bar{y}) \in D \text{ such that } x \geq \bar{x} \text{ and } 0 \leq y \leq \bar{y}\}.$$

Since X is comprehensive we have $0 \in X$ and $(0, h(0)) \in D$. Together with the definition of Ω this implies $(0,0) \in \Omega$. Closedness and convexity of Ω are immediate consequences of compactness and convexity of D. The monotonicity requirements for Ω are easily verified so that B1 follows. Since V and h are continuous functions it follows that $V(z) - \rho V(h(z))$ is continuous with respect to $z \in X$. Because X is compact, the number

$$M = \min\{V(z) - \rho V(h(z)) \mid z \in X\} \tag{13}$$

is a well defined and finite real number. For every $z \in \operatorname{int} X$ we define the function F_z: $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ by (9) where $p_z \geq 0$ and $q_z \geq 0$ are the subgradients

[10] The notation $y \gg 0$ means that all components of the vector $y \in \mathbb{R}^n$ are strictly positive.

of V whose existence is ensured by $R^+(\rho)$ and the interiority of z. We claim that $F_z(x,y) \geq M$ holds for all $z \in \text{int } X$ and all $(x,y) \in \Omega$. To prove the claim consider any fixed pair $(x,y) \in \Omega$. There exists a pair $(\bar{x},\bar{y}) \in D$ such that $x \geq \bar{x}$ and $0 \leq y \leq \bar{y}$. Moreover, because D is the convex hull of the graph of h, it follows that there exist nonnegative real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ and elements $x_1, x_2, \ldots, x_m$ of X such that $\sum_{i=1}^m \lambda_i = 1$ and $\sum_{i=1}^m \lambda_i (x_i, h(x_i)) = (\bar{x}, \bar{y})$. Since $p_z \geq 0$ and $q_z \geq 0$ we have

$$(14) \qquad p_z x - \rho q_z y \geq p_z \bar{x} - \rho q_z \bar{y} = \sum_{i=1}^{m} \lambda_i [p_z x_i - \rho q_z h(x_i)].$$

Using $R^+(\rho)$ we obtain

$$V(h(z)) - V(h(x_i)) + q_z[h(x_i) - h(z)]$$
$$\leq (1/\rho)[V(z) - V(x_i) + p_z(x_i - z)]$$

for all i. This inequality can be rewritten as

$$V(x_i) - \rho V(h(x_i)) \leq V(z) - \rho V(h(z)) + [p_z x_i - \rho q_z h(x_i)]$$
$$+ [\rho q_z h(z) - p_z z].$$

Multiplying this inequality by λ_i and summing over all i, we get

$$\sum_{i=1}^{m} \lambda_i [V(x_i) - \rho V(h(x_i))]$$
$$\leq V(z) - \rho V(h(z)) + \sum_{i=1}^{m} \lambda_i [p_z x_i - \rho q_z h(x_i)]$$
$$+ [\rho q_z h(z) - p_z z].$$

Using this together with (13) and (14), it follows that

$$M \leq \sum_{i=1}^{m} \lambda_i [V(x_i) - \rho V(h(x_i))] \leq V(z) - \rho V(h(z)) + (p_z x - \rho q_z y)$$
$$+ [\rho q_z h(z) - p_z z] = F_z(x,y).$$

Thus we have proved that $F_z(x,y) \geq M$ for all $z \in \text{int } X$ and all $(x,y) \in \Omega$. Now define $U(x,y) = \inf\{F_z(x,y) \mid z \in \text{int } X\}$. From this definition and from the facts we have already proved it follows that $M \leq U(x,y) \leq F_z(x,y)$ for all $(x,y) \in \Omega$ and all $z \in \text{int } X$. Since F_z is bounded on Ω, it follows that U is bounded, too. Because U is the infimum of affine functions it is concave and upper semi-continuous. It remains to verify the monotonicity properties required in B2. To this end note that p_z and q_z are nonnegative vectors. This implies that $F_z(x,y)$ is nondecreasing with respect to x and nonincreasing with respect to y. These monotonicity properties of F_z are obviously inherited by U such that the verification of B2 is complete. Assumption A3 holds trivially.

STEP 2: Continuity of h and V as well as strict concavity and monotonicity of V hold by assumption $R^+(\rho)$. The property $h(x) \in \Omega_x$ for all $x \in X$ holds by construction of Ω. Thus C1 and C2 of Proposition 2 are satisfied. It remains to verify C3. We first consider the case $x \in \operatorname{int} X$ and then the case where x is a boundary point of X.

STEP 2(a): If $x \in \operatorname{int} X$, then the verification of (4) and (5) is identical with Step 2 in the proof of Theorem 2.

STEP 2(b): Let x be a boundary point of X and let $y \in \Omega_x$. Furthermore, let $\bar{x} \in \operatorname{int} X$ and $\bar{y} \in \Omega_{\bar{x}}$. For every $\lambda \in (0,1)$ define $x_\lambda = (1-\lambda)\bar{x} + \lambda x$ and $y_\lambda = (1-\lambda)\bar{y} + \lambda y$. Convexity of Ω implies that $(x_\lambda, y_\lambda) \in \Omega$. Moreover, we have $x_\lambda \in \operatorname{int} X$ for all $\lambda \in (0,1)$ and $\lim_{\lambda \nearrow 1}(x_\lambda, y_\lambda) = (x, y)$. From Step 2(a) we know that

$$U(x_\lambda, y_\lambda) + \rho V(y_\lambda) \leq V(x_\lambda) = U(x_\lambda, h(x_\lambda)) + \rho V(h(x_\lambda)) \tag{15}$$

for all $\lambda \in (0,1)$. Continuity of V and h implies that $\lim_{\lambda \nearrow 1} V(x_\lambda) = V(x)$ and $\lim_{\lambda \nearrow 1} V(y_\lambda) = V(y)$. Upper semi-continuity and concavity of U show that $\lim_{\lambda \nearrow 1} U(x_\lambda, y_\lambda) = U(x, y)$ (see, e.g., Corollary 7.5.1 in Rockafellar (1970)). In the limit as λ approaches 1, the inequality on the left-hand side of (15) implies therefore that

$$V(x) \geq U(x, y) + \rho V(y) \tag{16}$$

holds. Continuity of V and h and upper semi-continuity of U imply furthermore that $V(h(x)) = \lim_{\lambda \nearrow 1} V(h(x_\lambda))$ and $U(x, h(x)) \geq \limsup_{\lambda \nearrow 1} U(x_\lambda, h(x_\lambda))$. Thus, in the limit as λ approaches 1 the equation on the right-hand side of (15) yields

$$V(x) \leq U(x, h(x)) + \rho V(h(x)). \tag{17}$$

From (16) and (17) it follows immediately that (5) holds. The right-hand side of (16) is a strictly concave function with respect to y which attains its maximum over Ω_x at a unique point y. Since we know from (17) that this point is $y = h(x)$ the strict inequality must hold in (16) for all $y \in \Omega_x$ that are different from $h(x)$. This establishes (4). We have therefore verified conditions C1–C3 of Proposition 2 and it follows that (Ω, U, ρ) satisfies A4 and that (h, V) is its solution. *Q.E.D.*

Theorem 4 shows that $R^+(\rho)$ is almost a necessary and sufficient condition for the rationalizability of a pair (h, V) by a dynamic optimization problem satisfying B1, B2, A3, and A4. There is only one additional requirement in the necessity part of the theorem, namely that there exists $(x, y) \in \Omega$ such that $y \gg 0$. This condition, however, is satisfied by any dynamic optimization problem (Ω, U, ρ) which rationalizes (h, V) provided that h is nondegenerate. The following lemma makes precise what we mean by nondegenerate.[11]

[11] We denote the ith component of a vector $z \in \mathbb{R}^n$ by $z_{(i)}$, that is, $z = (z_{(1)}, z_{(2)}, \ldots, z_{(n)})$.

LEMMA 1: *Let $X \subseteq \mathbb{R}^n_+$ be a compact, convex, and comprehensive set with nonempty interior and let h: $X \mapsto X$ be a given function. Assume that for every $i \in \{1,2,\dots,n\}$ there exists $x_i \in X$ such that $h_{(i)}(x_i) > 0$ and that h is rationalized by a dynamic optimization problem (Ω, U, ρ) satisfying $B1$, $B2$, $A3$, and $A4$. Then it follows that there exists $(x,y) \in \Omega$ such that $y \gg 0$.*

PROOF: By assumption we must have $(x_i, h(x_i)) \in \Omega$ for all $i \in \{1,2,\dots,n\}$. Since Ω is a convex set it follows that $(x,y) = (1/n)\Sigma_{i=1}^n (x_i, h(x_i)) \in \Omega$. Because $h(x_i) \geq 0$ and $h_{(i)}(x_i) > 0$ for all i, it follows that $y_{(i)} \geq h_{(i)}(x_i)/n > 0$. Therefore, we have $y \gg 0$ and the proof is complete. *Q.E.D.*

In the optimal growth context, the assumption that for all i there exists x_i such that $h_{(i)}(x_i) > 0$ means that every good is actually produced in some state. It does not necessarily imply that there exists a state in which all goods are produced simultaneously. As an application of Theorem 4 let us consider the standard aggregative model of optimal growth. This is a dynamic optimization model (Ω, U, ρ) on X satisfying B1, B2, A3, and A4 such that the following is true: $n = 1$, $X = [0, b]$ for some $b > 0$, and there exists $(x,y) \in \Omega$ with $y > 0$. We now show that a pair (h, V) can be rationalized by a standard aggregative model of optimal growth if and only if $R^+(\rho)$ holds.

THEOREM 5: *Let $X = [0, b]$ with $b > 0$ and let h: $X \mapsto X$ and V: $X \mapsto \mathbb{R}$ be given functions. There exists a standard aggregative model of optimal growth with discount factor $\rho \in (0,1)$ that rationalizes (h, V) if and only if condition* $R^+(\rho)$ *holds.*

PROOF: (a) Assume that (h, V) can be rationalized by a standard aggregative model of optimal growth (Ω, U, ρ). Since all the conditions of Theorem 4a are satisfied, it follows that $R^+(\rho)$ holds. (b) Assume that (h, V) satisfies $R^+(\rho)$ for some $\rho \in (0,1)$. We consider two cases.

CASE 1: If $h(x) = 0$ for all $x \in X$, then there exists $z \in \operatorname{int} X$ such that $h(z) = 0$. Since z is in the interior of X we know that V is subdifferentiable at z. Condition $R^+(\rho)$ then implies that V is also subdifferentiable at $h(z) = 0$. Thus, there exists a nonnegative number q such that

$$V(y) < V(0) + qy \qquad (18)$$

for all $y \in X \setminus \{0\}$. Now define $\Omega = X \times X$ and $U(x,y) = V(x) - \rho[V(0) + qy]$. Assumption B1 is obviously satisfied and so is the additional requirement that there exists $(x,y) \in \Omega$ with $y > 0$. The utility function U is continuous and concave and, since q is nonnegative and V is nondecreasing, U is nondecreasing with respect to x and nonincreasing with respect to y. Thus, B2 is satisfied. Assumption A3 holds trivially. To show that (Ω, U, ρ) rationalizes (h, V) it is therefore sufficient to verify the conditions of Proposition 2. C1 and C2 follow immediately from $R^+(\rho)$. Condition C3, on the other hand, is a simple consequence of the definition of U, the inequality in (18), and $h(x)=0$. Thus, Proposition 2 implies that A4 holds and that (Ω, U, ρ) rationalizes (h, V).

CASE 2: If there exists $x \in X$ and $h(x) > 0$, then the result follows immediately from Theorem 4b and from Lemma 1. *Q.E.D.*

5. CONCLUDING REMARKS

In this paper we have derived necessary and sufficient conditions for a pair of functions to be the optimal policy function and the optimal value function of a dynamic optimization problem satisfying the standard assumptions of optimal growth theory. We have also demonstrated that the gap between the necessary and the sufficient condition is very small, and how it can be completely eliminated if one imposes free disposal and monotonicity assumptions.

Most of the rationalizability conditions for dynamic optimization problems that are available in the literature can easily be derived from our conditions. Our results can also be used to derive tight discount factor restrictions for the occurrence of complicated dynamics in optimization models. The first contribution in this direction came from Sorger (1992) and the most powerful results obtained so far are those by Mitra (1996b), Montrucchio and Sorger (1996), and Nishimura and Yano (1996).[12] Two examples that have been considered by many contributors to this literature are the logistic map $h_1(x) = 4x(1-x)$ and the tent map $h_2(x) = 1 - |2x - 1|$, both defined on the unit interval. These maps are standard examples of chaotic dynamics. Using the rationalizability conditions of the present paper it is possible to prove that h_1 can be rationalized by a model (Ω, U, ρ) satisfying A1–A6 if and only if $\rho \in (0, 1/16]$, and that h_2 can be rationalized by a model (Ω, U, ρ) satisfying A1–A4 if and only if $\rho \in (0, 1/4]$. For proofs of these results and several other applications of the conditions derived in the present paper we refer to Mitra and Sorger (1997).

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